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CONTINUITY AND DIFFERENTIABILITY

IMPORTANT FORMULAE

1. Continuous Function :

If $\underset{h\to 0}{\text{Lt}}\ f(a-h) = \underset{h\to 0}{\text{Lt}}\ f(a+h) = f(a)$ then $f(x)$ is said to be continuous at $x = a$.

- A function $f(x)$ is said to be continuous at $x = a$ if $f(x)$ is defined at $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to 0} f(x) = f(a)$.
- A function $f(x)$ is said to be continuous at a if $\lim_{x\to a-0} f(x) = \lim_{x\to a+0} f(x) = f(a)$.
- **Geometrical meaning of continuity :** In geometrical form, a function $f(x)$ is continuous at $x = a$ if the graph of $y = f(x)$ has no break at $x = a$.

2. Differentiability :

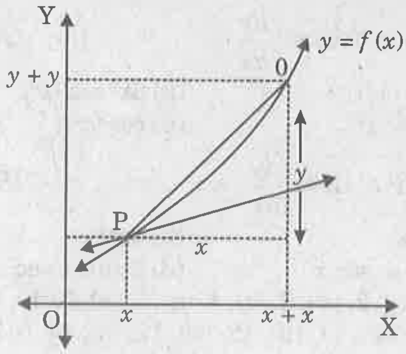
$$\text{LHD} = f'(a-0) = \lim_{x\to a-0} \frac{f(x)-f(a)}{x-a},$$

$$\text{RHD} = f'(a+0) = \lim_{x\to a-0} \frac{f(x)-f(a)}{x-a},$$

If $f'(a-0) = f'(a+0) = k$, then $f(x)$ is differentiable at $x = a$ and we write that $f'(a) = k$.

- **Relation between continuity and Differentiability :**
  1. If the function $f(x)$ is differentiable at any point, then it is also continuous at that point.
  2. If a function $f(x)$ is not differentiable at any point then it may or may not be continuous at that point.
  3. If a function is continuous at any point, then it may or may not be differentiable at that point.
  4. If a function is discontinuous at any point, then it is not differentiable at that point.
  5. (i) If $f'(a-0)$ is a definite number, then $f(x)$ will be continuous from L.H.S.

     (ii) If $f'(a+0)$ is a definite number, then $f(x)$ will be continuous from R.H.S.
- **Geometrical meaning of Derivative at a Point :** In geometrical form, $f(x)$ is differentiable at $x = a$ if a tangent line can draw at $x = a$ on the graph of $y = f(x)$.



Important Rule :

- $y = c.f(x) \Rightarrow \frac{dy}{dx} = cf'(x)$

  $y = f(x) \pm g(x) \Rightarrow \frac{dy}{dx} = f'(x) \pm g'(x)$
- **Product Rule**

  $$y = f(x).g(x) \Rightarrow \frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$

  $$y = u(x).v(x).w(x)$$

  $$\Rightarrow \frac{dy}{dx} = u'(x).v(x).w(x) + u(x).v'(x)w(x) + u(x).v(x).w'(x)$$
- **Division Rule**

  $$y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$
- **Chain Rule :** If $y = f(g(x))$, *i.e.*, $y = f(t)$, where $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$.
- $\frac{d}{dx}\{{}_a f(x)\} = {}_a f(x) \log_e^a f'(x); \frac{d}{dx}\{\log f(a)\} = \frac{f'(x)}{f(x)}$

  (i) $\frac{dy}{dx}(e^x) = e^x$

  (ii) $\frac{d}{dx}(a^x) = a^x \log_e a$

  (iii) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

  (iv) $\frac{d}{dx}(e^z) = \frac{d}{dz}(e^z)\frac{dz}{dx}$, where z is a function of x.

  (v) $\frac{d}{dx}(a^z) = \frac{d}{dz}(a^z).\frac{dz}{dx}$, where z is a function of x.

  (vi) $\frac{d}{dx}(\log z) = \frac{d}{dz}(\log z).\frac{dz}{dx}$, where z is a function of x.

  Derivative of the function of the form of $a^{f(x)}$ and $\log(f(x))$, where $f(x)$ is a function of x.
- **Implicit derivative :** If $y = f(t)$ and $x = f(t)$, where t is a parameter, then

  $$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
- **Rolle's Theorem :**

  Statement : If a function $f(x)$

  (i) is continuous in closed interval $[a, b]$ or is continuous at every point of interval $[a, b]$.

(ii) is differentiable in open interval (a, b) or is differentiable at every point of interval (a, b).

(iii) and $f(a) = f(b)$, then there will exist atleast one point x, where $a < x < b$ such that $f'(c) = 0$.

- **Lagrange's Mean Value Theorem :**

Statement : If any function $f(x)$

(i) is continuous in closed interval $[a, b]$ or continuous at each point in the interval $[a, b]$.

(ii) is differentiable in open interval (a, b) or differentiable at each point in the interval (a, b), then there exists atleast one c, where $a < c < b$ such that :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **Differential Table :**

Function	*Differential*	*Function*	*Differential*
$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$	$y = e^x$	$\frac{dy}{dx} = e^x$
$y = c$	$\frac{dy}{dx} = 0$	$y = a^x$	$\frac{dy}{dx} = a^x.\log_e a$
$y = x$	$\frac{dy}{dx} = 1$	$y = \log x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \frac{1}{x}$	$\frac{dy}{dx} = \frac{-1}{x^2}$	$y = \log_a x$	$\frac{dy}{dx} = \frac{1}{x \log_e a}$
$y = \sqrt{x}$	$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$	$y = \sin^{-1} x$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$
$y = \sin x$	$\frac{dy}{dx} = \cos x$	$y = \cos^{-1} x$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$	$y = \tan^{-1} x$	$\frac{dy}{dx} = \frac{1}{1+x^2}$
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x$	$y = \cot^{-1} x$	$\frac{dy}{dx} = \frac{-1}{1+x^2}$
$y = \cot x$	$\frac{dy}{dx} = -\text{cosec}^2 x$	$y = \sec^{-1} x$	$\frac{dy}{dx} = \frac{1}{\|x\|\sqrt{x^2-1}}$
$y = \sec x$	$\frac{dy}{dx} = \sec x \tan x$	$y = \text{cosec}^{-1} x$	$\frac{dy}{dx} = \frac{-1}{\|x\|\sqrt{x^2-1}}$
$y = \text{cosec}\, x$	$\frac{dy}{dx} = -\text{cosec}\, x \cot x$		

Multiple Choice Questions

1. If $p(x) = \begin{cases} \frac{x}{15} : x = 1, 2, 3, 4, 5 \\ 0 : \text{otherwise} \end{cases}$, then $p(x = 1)$ is : (*BSEB*, 2011)

(a) $\frac{1}{15}$ (b) $\frac{2}{15}$

(c) $\frac{1}{5}$ (d) none of these

2. If $y = \log x^x$, then $\frac{dy}{dx} =$ (*BSEB*, 2010)

(a) 1 (b) $\log x$

(c) $\log (e^x)$ (d) none of these

3. If $y = e^{2x}$, then $\frac{d^2y}{dx^2} =$

(a) $2y$ (b) y (c) $4y$ (d) $8y$

4. If $y = \tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}$, then $\frac{dy}{dx} =$

(a) $\frac{1}{2(1+x^2)}$ (b) $\frac{1}{1+x^2}$

(c) $\frac{2}{1+x^2}$ (d) none of these

5. If $\sqrt{x} + \sqrt{y} = 5$, then at $(4, a)$, $\frac{dy}{dx} =$

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $-\frac{3}{2}$ (d) $-\frac{2}{3}$

6. If $x = at^2$, $y = 2at$, then $\frac{dy}{dx} =$

(a) t (b) $\frac{1}{t}$ (c) at (d) $\frac{a}{t}$

7. A function f is said to be continuous at $x = a$, if :

(a) $\lim_{x \to a}$ exists (b) $\lim_{x \to a}$ does not exist

(c) $f(a)$ exists (d) none of these

8. The function f defined by $f(x) = \begin{cases} \frac{x^3 - 8}{x - 2}, x \neq 2 \\ 12, x = 2 \end{cases}$ is :

(a) not continuous at $x = 2$

(b) continuous at $x = 2$

(c) continuous at $x = 3$

(d) not continuous at $x = -2$

9. If $y = \sqrt{x + \sqrt{x + \sqrt{x + ... \infty}}}$, then $\frac{dy}{dx} =$

(a) $\frac{1}{2y+1}$ (b) $\frac{1}{2y-1}$ (c) $\frac{1}{y+1}$ (d) $\frac{1}{y-1}$

10. If Rolle's Theorem is true on [1, 5] for the function $f(x) = x^2 - 6x + 5$, then the value of c is :

(a) 1 (b) 2 (c) 3 (d) 4

11. $\frac{d}{dx}(\sin^{-1} x) =$ (*BSEB*, 2015)

(a) $\frac{1}{\sqrt{1-x^2}}$ (b) $-\frac{1}{\sqrt{1-x^2}}$ (c) $2(1-x^2)$ (d) $(1-x^2)$

12. $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) =$ (*BSEB*, 2015)

(a) 0 (b) 1 (c) $\frac{\pi}{2}$ (d) $\frac{1}{\sqrt{1-x^2}}$

13. If $y = \sin(x^3)$, then $\frac{dy}{dx} =$ (*BSEB*, 2015)

(a) $x^3 \cos(x^3)$ (b) $3x^2 \sin(x^3)$

(c) $3x^2 \cos(x^3)$ (d) $\cos(x^3)$

14. If $y = \tan^2 x$, then $\frac{dy}{dx} =$ (*BSEB*, 2015)

(a) $\sec^2 x$ (b) $\sec^4 x$

(c) $2 \tan x. \sec x$ (d) $2 \tan x \sec^2 x$

Ans. 1. (a), 2. (d), 3. (c), 4. (a), 5. (c), 6. (b), 7. (d), 8. (b), 9. (b), 10. (c), 11. (a), 12. (a), 13. (b), 14. (d).

➠ Very Short Answer Type Questions

Q. 1. Find $\frac{d}{dx}(\sin\sqrt{x})$. (*BSEB*, 2014)

Solution :

$$\frac{d}{dx}(\sin\sqrt{x}) = \cos\sqrt{x}\frac{d}{dx}(\sqrt{x}) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

Q. 2. Find $\frac{d}{dx}(\sin^2 x)$. (*BSEB*, 2014)

Solution :

$$\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}(\sin x)^2 = 2\sin x\frac{d}{dx}(\sin x) = 2\sin x\cos x = \sin 2x$$

Q. 3. Find $y = \frac{1}{\sin x} + e^x$, then find $\frac{dy}{dx}$. (*BSEB*, 2014)

Solution :

$$y = \frac{1}{\sin x} + e^x$$

$\Rightarrow$ $$y = \text{cosec } x + e^x$$

$\therefore$ $$\frac{dy}{dx} = -\text{cosec } x\cot x + e^x$$

Q. 4. If $y = \sqrt{x^2 + ax + 1}$, then find $\frac{dy}{dx}$. (*BSEB*, 2014)

Solution :

$$y = \sqrt{x^2+ax+1}$$

$\Rightarrow$ $$y = (x^2+ax+1)^{1/2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2}(x^2+ax+1)^{1/2-1}\frac{d}{dx}(x^2+ax+1) = \frac{2x+a}{2\sqrt{x^2+ax+1}}$$

Q. 5. Write the derivative of $\sin x$ w.r.t. $\cos x$. [*CBSE*, 2014 (*Comptt.*)]

Solution :

Let $u = \sin x$

then $\frac{du}{dx} = \cos x$

and let $v = \cos x$

then $\frac{dv}{dx} = -\sin x$

$\therefore$ $$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\cos x}{-\sin x} = -\cot x$$

Q. 6. Prove that : (*BSER*, 2013)

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right] = \sqrt{a^2-x^2}$$

Solution :

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{2}\left[\sqrt{a^2-x^2} + x.\frac{1}{2\sqrt{a^2-x^2}}.(-2x)\right] + \frac{a^2}{2}\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{1}{a}$$

$$= \frac{1}{2}\sqrt{a^2-x^2} - \frac{x^2}{2\sqrt{a^2-x^2}} + \frac{a^2}{2\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\sqrt{a^2-x^2} + \frac{a^2-x^2}{2\sqrt{a^2-x^2}}$$

$$= \frac{1}{2}\sqrt{a^2-x^2} + \frac{1}{2}\sqrt{a^2-x^2}$$

$$= \sqrt{a^2-x^2}$$

Q. 7. If $x = a\cos\theta$ and $y = b\sin\theta$, then find $\frac{dy}{dx}$. (*BSEB*, 2014)

Solution :

$$x = a\cos\theta$$

$\Rightarrow$ $$\frac{dx}{d\theta} = -a\sin\theta$$

and $$y = b\sin\theta$$

$\Rightarrow$ $$\frac{dy}{d\theta} = b\cos\theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

Q. 8. If $y = \sin(\cot x)$, then find $\frac{dy}{dx}$. (*JAC*, 2013)

Solution :

$$y = \sin(\cot x)$$

$\Rightarrow$ $$\frac{dy}{dx} = \cos(\cot x)\frac{d}{dx}(\cot x) = \cos(\cot x) - (-\text{cosec}^2 x) = -\text{cosec}^2 x\cos(\cot x)$$

Q. 9. If $f(x) = 4x^2 - 5x$, then find the value of $\frac{f(x+h)-f(x)}{h}$. (*JAC*, 2013)

Solution :

$$f(x) = 4x^2 - 5x$$

$\therefore$ $$f(x+h) = 4(x+h)^2 - 5(x+h) = 4x^2 + 4h^2 + 8xh - 5x - 5h$$

$\therefore$ $$f(x+h) - f(x) = (4x^2 + 4h^2 + 8xh - 5x - 5h) - (4x^2 - 5x) = 4h^2 + 8xh - 5h = h(4h + 8x - 5)$$

$\therefore$ $$\frac{f(x+h)-f(x)}{h} = 4h + 8x - 5$$

Q. 10. If $y = Pe^{ax} + Qe^{bx}$, show that : (*AI CBSE*, 2014)

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$$

Solution :

$$y = Pe^{ax} + Qe^{bx}$$

$$\Rightarrow \quad \frac{dy}{dx} = aPe^{ax} + bQe^{bx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = a^2Pe^{ax} + b^2Qe^{bx}$$

$$\therefore \quad \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = a^2Pe^{ax} + b^2Qe^{ax} - (a+b)(aPe^{ax} + bQe^{bx}) + ab(Pe^{ax} + Qe^{bx})$$

$$= a^2Pe^{ax} + b^2Qe^{ax} - a^2Pe^{ax} - abQe^{bx} - abPe^{ax} - b^2Qe^{bx} + abPe^{ax} + abQe^{bx}$$

$$= 0$$

➠ Short Answer Type Questions

Q. 1. Find $\frac{dy}{dx}$, if : (*J.A.C.*, 2011, 13)

$$y = \frac{1-\cos x}{1+\cos x}$$

Solution :

If $y = \frac{1-\cos x}{1+\cos x}$

$$\frac{dy}{dx} = \frac{(1+\cos x)\frac{d}{dx}(1-\cos x) - (1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)(\sin x) - (1-\cos x)(-\sin x)}{(1+\cos x^2)}$$

$$= \frac{\sin x + \sin x\cos x + \sin x - \sin x\cos x}{(1+\cos x)^2}$$

$$= \frac{2\sin x}{(1+\cos x)^2}$$

Q. 2. If $y = \sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}(1-x^2)\right]$, find $\frac{dy}{dx}$. (*BSEB*, 2014)

Solution :

$$y = \sin^{-1}\left[x\sqrt{1-x} - \sqrt{x}(1-x^2)\right]$$

Put $x = \sin\theta$ and $\sqrt{x} = \sin\phi$

$$y = \sin^{-1}[\sin\theta\sqrt{1-\sin^2\phi} - \sin\phi\sqrt{1-\sin^2\theta}]$$

$$\Rightarrow \quad y = \sin^{-1}[\sin\theta\cos\phi - \sin\phi\cos\theta]$$

$$\Rightarrow \quad y = \sin^{-1}\sin(\theta - \phi)$$

$$\Rightarrow \quad y = \theta - \phi$$

$$\Rightarrow \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x) - \frac{d}{dx}(\sin^{-1}\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\,\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

Q. 3. If $y = \log\left[x + \sqrt{x^2+a^2}\right]$, show that :

$$(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0 \qquad (CBSE, 2013)$$

Solution :

$$y = \log(x + \sqrt{x^2+a^2})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}}\left\{1 + \frac{1}{2}\frac{2x}{\sqrt{x^2+a^2}}\right\}$$

$$= \frac{1}{x+\sqrt{x^2+a^2}}\,\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}$$

$$= \frac{1}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \sqrt{x^2+a^2}\,\frac{dy}{dx} = 1$$

Squaring, we get

$$(x^2+a^2)\left(\frac{dy}{dx}\right)^2 = 1$$

Differentiating w.r.t. x, we get

$$(x^2+a^2)\,2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 0$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Q. 4. If $y^x = e^{y-x}$, prove that : (*AI CBSE*, 2013)

$$\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$

Solution :

$$y^x = e^{y-x} \qquad ...(1)$$

Taking logarithm on both sides,

$$x\log y = (y-x)\log e$$

$$\Rightarrow \quad x\log y = y - x \qquad ...(2)$$

Differentiating w.r.t. x,

$$x.\frac{1}{y}.\frac{dy}{dx} + \log y.1 = \frac{dy}{dx} - 1$$

$$\Rightarrow \quad \left(\frac{x}{y} - 1\right)\frac{dy}{dx} = -(1+\log y)$$

$$\Rightarrow \quad \frac{x-y}{y}\frac{dy}{dx} = -(1+\log y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y(1+\log y)}{x-y}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y(1+\log y)}{-x\log y} \qquad \text{[from (2)]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\,\frac{1+\log y}{\log y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y} \qquad \text{[From (2)]}$$

$[\because x(1+\log y) = y]$

$$\Rightarrow \quad \frac{y}{x} = 1 + \log y$$

Q. 5. Test the continuity of the function $f(x)$ at $x = 0$, where (*J.A.C.*, 2010; 13)

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

Solution :

Given $\quad f(x) = x \sin \dfrac{1}{x}$, when $x \neq 0$

$\therefore$ when $x < 0$, $\quad f(x) = x \sin \dfrac{1}{x}$

and when $x > 0$, $\quad f(x) = x \sin \dfrac{1}{x}$

Given $\quad f(0) = 0$

$$\begin{aligned} \text{L.H.L.} &= \lim_{x \to 0^-} f(x) \\ &= \lim_{h \to 0} f(0-h) \\ &= \lim_{h \to 0} f(-h) \\ &= \lim_{h \to 0} \left\{(-h) \sin \left(\frac{1}{-h}\right)\right\} \\ &= \lim_{h \to 0} \left(h \sin \frac{1}{h}\right) \\ &= 0 \end{aligned}$$

$$\left(\because -1 \leq \sin \frac{1}{h} \leq 1, \therefore \lim_{h \to 0} \sin \frac{1}{h} = 0\right)$$

$$\begin{aligned} \text{R.H.L.} &= \lim_{h \to 0^+} f(x) \\ &= \lim_{h \to 0} f(0+h) \\ &= \lim_{h \to 0} f(h) \\ &= \lim_{h \to 0} h \sin \frac{1}{h} \\ &= 0 \end{aligned}$$

$$\left(\because -1 \leq \sin \frac{1}{h} \leq 1, \therefore \lim_{h \to 0} \sin \frac{1}{h} = 0\right)$$

$\because \quad$ L.H.L. = R.H.L. = $f(0)$

$\therefore f(x)$ is continuous at $x = 0$.

Q. 6. If $y = \tan^{-1}\left(\dfrac{a}{x}\right) + \log \sqrt{\dfrac{x-a}{x+a}}$, prove that :

$$\frac{dy}{dx} = \frac{2a^3}{x^4 - a^4}$$

[*AI CBSE*, 2014 (*Comptt.*)]

Solution :

$$\begin{aligned} y &= \tan^{-1}\left(\frac{a}{x}\right) + \log \sqrt{\frac{x-a}{x+a}} \\ &= \tan^{-1}\left(\frac{a}{x}\right) + \frac{1}{2} \log (x-a) - \frac{1}{2} \log (x+a) \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1 + \left(\frac{a}{x}\right)^2}\left(-\frac{a}{x^2}\right) + \frac{1}{2} \cdot \frac{1}{x-a} - \frac{1}{2} \cdot \frac{1}{x+a}$$

$$\begin{aligned} &= -\frac{a}{x^2+a^2} + \frac{1}{2}\left(\frac{1}{x-a} - \frac{1}{x+a}\right) \\ &= -\frac{a}{x^2+a^2} + \frac{1}{2}\left\{\frac{(x+a)-(x-a)}{x^2-a^2}\right\} \\ &= -\frac{a}{x^2+a^2} + \frac{a}{x^2-a^2} \\ &= \frac{-a(x^2-a^2) + a(x^2+a^2)}{x^4-a^4} \\ &= \frac{a[-x^2+a^2+x^2+a^2]}{x^4-a^4} \\ &= \frac{2a^3}{x^4-a^4} \end{aligned}$$

Proved

Q. 7. If $(x-y)e^{\frac{x}{x-y}} = a$, Prove that :

$$y\frac{dy}{dt} + x = 2y$$

[*CBSE*, 2014 (*Comptt.*)]

Solution :

$$(x-y)e^{\frac{x}{x-y}} = a$$

Taking log of both sides, we get

$$\log(x-y) + \frac{x}{x-y} = \log a$$

Differentiating w.r.t. x, we get

$$\frac{1}{x-y}\left(1 - \frac{dy}{dx}\right) + \frac{(x-y).1 - x\left(1 - \frac{dy}{dx}\right)}{(x-y)^2} = 0$$

$$\Rightarrow (x-y)\left(1 - \frac{dy}{dx}\right) + (x-y) - x\left(1 - \frac{dy}{dx}\right) = 0$$

$$\Rightarrow (x-y) - (x-y)\frac{dy}{dx} + (x-y) - x + x\frac{dy}{dx} = 0$$

$$\Rightarrow (x-2y) + y\frac{dy}{dx} = 0$$

$$\Rightarrow y\frac{dy}{dx} + x = 2y$$

Q. 8. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{6}$. (*AI CBSE*, 2013)

Solution :

$$x = a\cos^3\theta$$

$$\Rightarrow \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

and $\quad y = a\sin^3\theta$

$$\Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$= -\tan\theta$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(-\tan\theta)$$

$$= \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}$$

$$= \frac{\frac{d}{d\theta}(-\tan\theta)}{\frac{dx}{d\theta}}$$

$$= \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$= \frac{1}{3a\cos^4\theta\sin\theta}$$

at $\theta = \frac{\pi}{6}$, $$\frac{d^2y}{dx^2} = \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\frac{1}{2}}$$

$$= \frac{1}{3a.\frac{9}{16}.\frac{1}{2}} = \frac{32}{27a}$$

Q. 9. Find $\frac{dy}{dx}$, if $x^y + y^x = b^a + a^b$. (*BSER*, 2014)

Solution :

Let $u = x^y$

then $\log u = y\log x$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = y.\frac{1}{x} + \log x.\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{du}{dx} = u\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{du}{dx} = x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) \quad ...(1)$$

Let $v = y^x$

then $\log v = x\log y$

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y.1$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = y^x\left(\frac{y}{x}\frac{dy}{dx} + \log y\right) \quad ...(2)$$

Now $x^y + y^x = b^a + a^b$

$\Rightarrow \quad u + v = b^a + a^b$

$$\Rightarrow \quad \frac{du}{dx} + \frac{dv}{dx} = 0$$

$$\Rightarrow \quad x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + y^x\left(\frac{y}{x}\frac{dy}{dx} + \log y\right) = 0$$

$$\Rightarrow (x^y\log x + y^{x-1}x)\frac{dy}{dx} = -(yx^{y-1} + y^x\log y)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{yx^{y-1} + y^x\log y}{x^y\log x + y^{x-1}x}\right)$$

$$= \frac{-y\left(x^{y-1} + y^{x-1}\log y\right)}{x\left(x^{y-1}\log x + y^{x-1}\right)}$$

Q. 10. Differentiate the following function with respect to x. (*CBSE*, 2013)

$$(\log x)^x + x\log x$$

Solution :

$$y = (\log x)^x + x\log x$$

$\Rightarrow \quad y = u + v \quad ...(1)$

where $u = (\log x)^x$

and $v = x\log x$

Now $u = (\log x)^x$

$\Rightarrow \quad \log u = x\log(\log x)$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x.\frac{1}{\log x}.\frac{1}{x} + \log\log x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left(\frac{1}{\log x} + \log\log x\right)$$

$$\Rightarrow \quad \frac{du}{dx} = (\log x)^x\left\{\frac{1}{\log x} + \log\log x\right\} \quad ...(2)$$

$$v = x\log x$$

$$\Rightarrow \quad \frac{dv}{dx} = x.\frac{1}{x} + \log x.1$$

$$\Rightarrow \quad \frac{dv}{dx} = 1 + \log x \quad ...(3)$$

From (1), $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\log x)^x\left\{\frac{1}{\log x} + \log\log x\right\} + (1 + \log x)$$

Q. 11. If $x = \cos\theta$ and $y = \sin^3\theta$, then prove that :

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta(5\cos^2\theta - 1)$$

[*CBSE*, 2013; (*Comptt.*)]

Solution :

$$x = \cos\theta$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -\sin\theta$$

$$y = \sin^3\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\sin^2\theta\cos\theta}{-\sin\theta}$$

$$= -3\sin\theta\cos\theta = -\frac{3}{2}\sin 2\theta$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(-\frac{3}{2}\sin 2\theta\right)$$

$$= \frac{d}{d\theta}\left(-\frac{3}{2}\sin 2\theta\right)\frac{d\theta}{dx}$$

$$= \frac{\frac{d}{d\theta}\left(-\frac{3}{2}\sin 2\theta\right)}{\frac{dx}{d\theta}}$$

$$= \frac{-\frac{3}{2}.2.\cos 2\theta}{-\sin\theta}$$

$$= \frac{3\cos 2\theta}{\sin\theta}$$

$$\therefore y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \sin^3\theta\,\frac{3\cos 2\theta}{\sin\theta} + \frac{9}{4}\sin^2 2\theta$$

$$= 3\sin^2\theta\cos 2\theta + \frac{9}{4}.4\sin^2\theta\cos^2\theta$$

$$= 3\sin^2\theta\cos 2\theta + 9\sin^2\theta\cos^2\theta$$

$$= 3\sin^2\theta(\cos 2\theta + 3\cos^2\theta)$$

$$= 3\sin^2\theta(2\cos^2\theta - 1 + 3\cos^2\theta)$$

$$= 3\sin^2\theta(5\cos^2\theta - 1)$$

Q. 12. If $x = a\sin 2t\,(1+\cos 2t)$, $y = b\cos 2t\,(1-\cos 2t)$, prove that at $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{b}{a}$.

(JAC, A.I., CBSE, 2014)

Solution :

$$x = a\sin 2t\,(1+\cos 2t)$$

$$\Rightarrow \frac{dx}{dt} = a[\sin 2t(-2\sin 2t) + 2\cos 2t(1+\cos 2t)]$$

$$= 2a[\cos 2t + \cos^2 2t - \sin^2 2t]$$

$$= 2a(\cos 2t + \cos 4t) \quad ...(1)$$

$$y = b\cos 2t(1-\cos 2t)$$

$$\Rightarrow \frac{dy}{dt} = b[\cos 2t(2\sin 2t) + (1-\cos 2t)(-2\sin 2t)]$$

$$= 2b[-\sin 2t + 2\sin 2t\cos 2t]$$

$$= 2b[-\sin 2t + \sin 4t] \quad ...(2)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2b(-\sin 2t + \sin 4t)}{2a(\cos 2t + \cos 4t)}$$

at $t = \frac{\pi}{4}$,

$$\frac{dy}{dx} = \frac{2b\left(-\sin\frac{\pi}{2} + \sin\pi\right)}{2a\left(\cos\frac{\pi}{2} + \cos\pi\right)}$$

$$= \frac{2b(-1+0)}{2a(0-1)} = \frac{b}{a}$$

Q. 13. If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t(3 - 2\sin^2 t)$, find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. *(AI CBSE, 2014)*

Solution :

$$x = \cos t(3 - 2\cos^2 t)$$

$$\Rightarrow \frac{dx}{dt} = \cos t(4\cos t\sin t) - \sin t(3 - 2\cos^2 t)$$

$$\Rightarrow \frac{dx}{dt} = 4\sin t\cos^2 t - 3\sin t + 2\sin t\cos^2 t$$

$$\Rightarrow \frac{dx}{dt} = -3\sin t + 6\sin t\cos^2 t \quad ...(1)$$

and $$y = \sin t(3 - 2\sin^2 t)$$

$$\Rightarrow \frac{dy}{dt} = \cos t(3 - 2\sin^2 t) - 4\sin^2 t\cos t$$

$$\Rightarrow \frac{dy}{dt} = 3\cos t - 6\sin^2 t\cos t \quad ...(2)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3\cos t - 6\sin^2 t\cos t}{-3\sin t + 6\sin t\cos^2 t}$$

$$= \frac{3\cos t(1 - 2\sin^2 t)}{3\sin t(2\cos^2 t - 1)}$$

$$= \frac{3\cos t\cos 2t}{3\sin t\cos 2t}$$

$$= \cot t$$

at $t = \frac{\pi}{4}$,

$$\frac{dy}{dx} = \cot\frac{\pi}{4} = 1$$

Q. 14. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$, then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(CBSE, Delhi, Compartment, 2009, 12; BSEB, 2014)

Solution :

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right)$$

$$\Rightarrow \frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{a\cos^2 t}{\sin t}$$

$$\Rightarrow \frac{dy}{dx} = a\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{a\cos t}{a\cos^2 t} \times \sin t = \tan t$$

Now, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(\tan t)$$

$$= \frac{d}{dt}(\tan t)\frac{dt}{dx}$$

$$= \frac{\frac{d}{dt}(\tan t)}{\frac{dx}{dt}}$$

$$= \frac{\sec^2 t}{\left(\frac{a\cos^2 t}{\sin t}\right)} = \frac{\sin t}{a\cos^4 t}$$

Q. 15. If $(\tan^{-1} x)^y + y^{\cot x} = 1$, then find $\frac{dy}{dx}$.

[AI CBSE, 2014; (Comptt.)]

Solution :

Let $u = (\tan^{-1} x)^y$

then $\log u = y \log \tan^{-1} x$

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \frac{dy}{dx}\log\tan^{-1} x + \frac{y}{\tan^{-1} x}\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \frac{du}{dx} = u\left(\frac{dy}{dx}\log\tan^{-1} x + \frac{y}{\tan^{-1} x}\cdot\frac{1}{1+x^2}\right)$$

$$\Rightarrow \frac{du}{dx} = (\tan^{-1} x)^y\left\{\frac{dy}{dx}\log\tan^{-1} x + \frac{y}{\tan^{-1} x}\cdot\frac{1}{1+x^2}\right\} \quad ...(1)$$

Let $v = y^{\cot x}$

then $\log v = \cot x \log y$.

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = -\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left(-\operatorname{cosec}^2 x\log y + \frac{\cot x}{y}\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = y^{\cot x}\left\{-\operatorname{cosec}^2 x\log y + \frac{\cot x}{y}\frac{dy}{dx}\right\} \quad ...(2)$$

Now $(\tan^{-1} x)^y + y^{\cot x} = 1$

$$\Rightarrow \quad u + v = 1$$

$$\Rightarrow \quad \frac{du}{dx} + \frac{dv}{dx} = 0$$

$$\Rightarrow (\tan^{-1} x)^y\left\{\frac{dy}{dx}\log\tan^{-1} x + \frac{y}{\tan^{-1} x}\cdot\frac{1}{1+x^2}\right\}$$

$$+ y^{\cot x}\left\{-\operatorname{cosec}^2 x\log y + \frac{\cot x}{y}\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad -\left\{(\tan^{-1} x)^y\log\tan^{-1} x + y^{\cot x}\cdot\frac{\cot x}{y}\right\}\frac{dy}{dx}$$

$$= y^{\cot x}\operatorname{cosec}^2 x\log y - (\tan^{-1} x)^{y-1}\frac{y}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^{\cot x}\operatorname{cosec}^2 x\log y - (\tan^{-1} x)^{y-1}\frac{y}{1+x^2}}{(\tan^{-1} x)^y\log\tan^{-1} y + y^{\cot x}\frac{\cot y}{y}}$$

Q. 16. Differentiate the following with respect to x :

$$\sin^{-1}\left\{\frac{2^{x+1}3^x}{1+(36)^x}\right\}$$ *(AI CBSE, 2013)*

Solution :

$$y = \sin^{-1}\left\{\frac{2^{x+1}3^x}{1+(36)^x}\right\}$$

$$= \sin^{-1}\left\{\frac{2.2^x3^x}{1+(6^2)^x}\right\}$$

$$= \sin^{-1}\left\{\frac{2.6^x}{1+(6^x)^2}\right\}$$

Put $6^x = \tan\theta$.

then $y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow \quad y = 2\theta$$

$$\Rightarrow \quad y = 2\tan^{-1} 6^x$$

$$\Rightarrow \quad \frac{dy}{dx} = 2.\frac{1}{1+(6^x)^2}\frac{d}{dx}(6^x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2.6^x\log_e 6}{1+(36)^x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2^{x+1}3^x\log_e 6}{1+(36)^x}$$

Q. 17. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$. *(CBSE, 2014)*

Solution :

Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta$

then $u = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$

$$\Rightarrow \quad u = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$\Rightarrow \quad u = \tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad u = \theta = \sin^{-1} x$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad ...(1)$$

Let $v = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Put $x = \sin\theta$

then $v = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow$ $v = \sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow$ $v = \sin^{-1}\sin 2\theta$

$= 2\theta = 2\sin^{-1}x$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}} \qquad ...(2)$$

$$\therefore \quad \frac{dv}{dv} = \frac{du/dx}{dv/dx} = \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

Q. 18. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$. (*CBSE*, 2014)

Solution :

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta$

then $u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right)$

$= \tan^{-1}(\tan\theta/2)$

$= \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

$$\therefore \quad \frac{du}{dx} = \frac{1}{2(1+x^2)} \qquad ...(1)$$

Let $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

then $v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$$\therefore \quad \frac{dv}{dx} = \frac{2}{1+x^2} \qquad ...(2)$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{(1+x^2)}} = \frac{1}{4}$$

Q. 19. If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$. (*CBSE Delhi*, 2012; *JAC*, 2014)

Solution :

We have

$$(\cos x)^y = (\cos y)^x$$

Taking log on both sides, we get

$$y \log(\cos x) = x \log(\cos y)$$

Differentiating with respect to x,

$$y\left(\frac{1}{\cos x}\right)(-\sin x) + \frac{dy}{dx}\log(\cos x)$$

$$= x\left(\frac{1}{\cos y}\right)(-\sin y)\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow -y\tan x + \frac{dy}{dx}\log(\cos x)$$

$$= -x\tan y\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \frac{dy}{dx}[\log(\cos x) + x\tan y]$$

$$= y\tan x + \log(\cos y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\tan x + \log(\cos y)}{x\tan y + \log(\cos y)}$$

Q. 20. If $\sin y = x\sin(a+y)$, prove that :

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

(*CBSE Delhi*, 2012; *AI CBSE*, 2013)

Solution :

$$\because \quad \sin y = x\sin(a+y)$$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Hence Proved

Q. 21. If $y = (\tan^{-1}x)^2$, show that

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$$

(*CBSE, Delhi*, 2012 & *Outside Delhi*, 2012)

Solution :

We have

$$y = (\tan^{-1}x)^2$$

$$\frac{dy}{dx} = 2\tan^{-1}x.\frac{1}{1+x^2}$$

$\Rightarrow$ $$(1+x^2)\frac{dy}{dx} = 2\tan^{-1}x$$

Again differentiating on both sides, we get

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{2}{1+x^2}$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2.$$

Hence proved.

Q. 22. If $y = e^{m\sin^{-1}x}$, then prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$ *(BSER, 2013)*

Solution :

$$y = e^{m\sin^{-1}x}$$

$\Rightarrow$ $$\frac{dy}{dx} = e^{m\sin^{-1}x}\frac{m}{\sqrt{1-x^2}}$$

$\Rightarrow$ $$\sqrt{1-x^2}\frac{dy}{dx} = my$$

$\Rightarrow$ $$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$ (squaring)

Differentiating w.r.t. x, we get

$$(1-x^2)\frac{dy}{dx}\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 2m^2y\frac{dy}{dx}$$

cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$$

Q. 23. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

[*USEB*, 2013, *CBSE*, 14 (*Comptt.*)]

Solution :

$$x = a(\cos t + t\sin t)$$

$\Rightarrow$ $$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t$$

and $$y = a(\sin t - t\cos t)$$

$\Rightarrow$ $$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{at\sin t}{at\cos t} = \tan t$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(\tan t)$$

$$= \frac{d}{dt}(\tan t)\frac{dt}{dx}$$

$$= \frac{\frac{d}{dt}(\tan t)}{\frac{dx}{dt}}$$

$$= \frac{\sec^2 t}{at\cos t} = \frac{1}{at\cos^3 t}$$

at $t = \frac{\pi}{4}$,

$$\frac{d^2y}{dx^2} = \frac{1}{a.\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^3} = \frac{8\sqrt{2}}{a\pi}$$

Q. 24. If $x = \tan\left(\frac{1}{a}\log y\right)$, then show that $(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$

[*CBSE*, 2013 (*Comptt.*)]

Solution :

$$x = \tan\left(\frac{1}{a}\log y\right)$$

$\Rightarrow$ $$\frac{1}{a}\log y = \tan^{-1}x$$

$\Rightarrow$ $$\log y = a\tan^{-1}x$$

$\Rightarrow$ $$y = e^{a\tan^{-1}x}$$

$\Rightarrow$ $$\frac{dy}{dx} = e^{a\tan^{-1}x}\frac{a}{1+x^2}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{ya}{1+x^2}$$

$\Rightarrow$ $$(1+x^2)\frac{dy}{dx} = ya$$

$\Rightarrow$ $$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = a\frac{dy}{dx}$$

$\Rightarrow$ $$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$$

Q. 25. Find $\frac{dy}{dx}$, if $y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

[*CBSE* 2013, (*Comptt.*)]

Solution :

$$y = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$$

$\Rightarrow$ $$y = \sin^{-1}\left\{\frac{2.2^x}{1+(2^x)^2}\right\}$$

Put $2^x = \tan\theta$, then

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$\Rightarrow$ $y = \sin^{-1}(\sin 2\theta)$

$\Rightarrow$ $y = 2\theta$

$\Rightarrow$ $y = 2\tan^{-1}2^x$

$$\Rightarrow \quad \frac{dy}{dx} = 2\,\frac{1}{1+(2^x)^2}\,2^x \log_e 2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2^{x+1}\log_e 2}{1+4^x}$$

Q. 26. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$ (*CBSE*, 2014)

Solution :

$$y = x^x \quad \ldots(1)$$

Taking log on both sides, we get

$$\log y = x \log x$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \log x.1 = 1 + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = y\,(1 + \log x) \quad \ldots(2)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx}\,(1 + \log x) + \frac{y}{x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx}\,(1 + \log x) - \frac{y}{x} = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{dy}{dx}\,\frac{1}{y}\,\frac{dy}{dx} - \frac{y}{x} = 0 \quad \text{[From (2)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$

Q. 27. If $x = ae^\theta$ (sin θ – cos θ) and $y = ae^\theta$ (sin θ + cos θ), find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$. (*AI CBSE*, 2014)

Solution :

$$x = ae^\theta(\sin\theta - \cos\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = a\,[e^\theta(\cos\theta + \sin\theta) + e^\theta(\sin\theta - \cos\theta)]$$

$$\Rightarrow \quad \frac{dx}{d\theta} = 2\,ae^\theta \sin\theta \quad \ldots(1)$$

and $\quad y = ae^\theta(\sin\theta + \cos\theta)$

$$\Rightarrow \quad \frac{dy}{d\theta} = a\,[e^\theta(\cos\theta - \sin\theta) + e^\theta(\sin\theta + \cos\theta)]$$

$$\Rightarrow \quad \frac{dy}{d\theta} = 2\,ae^\theta \cos\theta \quad \ldots(2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2ae^\theta\cos\theta}{2ae^\theta\sin\theta} = \cot\theta$$

at $\theta = \frac{\pi}{4}$,

$$\frac{dy}{dx} = \cot\frac{\pi}{4} = 1$$

Q. 28. If $x = a \sin t$ and $y = a\left(\cos t + \log\tan\frac{t}{2}\right)$, find $\frac{d^2y}{dx^2}$. (*CBSE*, 2013)

Solution :

$$x = a\sin t$$

$$\Rightarrow \quad \frac{dx}{dt} = a\cos t \quad \ldots(1)$$

$$y = a\left(\cos t + \log\tan\frac{t}{2}\right)$$

$$\Rightarrow \quad \frac{dy}{dt} = a\left(-\sin t + \frac{\sec^2\frac{t}{2}}{\tan\frac{t}{2}}.\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{dy}{dt} = a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{1-\sin^2 t}{\sin t}\right) = \frac{a\cos^2 t}{\sin t} \quad \ldots(2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\frac{\cos^2 t}{\sin t}}{a\cos t} = \cot t \quad \ldots(3)$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy}{dx}\,(\cot t)$$

$$= \frac{d}{dt}\,(\cot t)\,\frac{dt}{dx} = \frac{\frac{d}{dt}(\cot t)}{\frac{dx}{dt}}$$

$$= -\frac{\operatorname{cosec}^2 t}{a\cos t} = -\frac{1}{a\sin^2 t\cot t}$$

➠ Long Answer Type Questions

Q. 1. Verify Rolle's Theorem for the function $f(x) = 2x^3 + x^2 - 4x - 2$ when $-\frac{1}{2} \le x \le \sqrt{2}$. (*BSEB*, 2014)

Solution :

We have

$$f(x) = 2x^3 + x^2 - 4x - 2$$

(1) ∵ $f(x)$ is a polynomial.

∴ $f(x)$ is continuous on R.

∴ In particular, $f(x)$ is continuous on $\left[-\frac{1}{2}, \sqrt{2}\right]$

(2) $f'(x) = 6x^2 + 2x - 4$

∵ $f'(x)$ exists uniquely on $\left(-\frac{1}{2}, \sqrt{2}\right)$

$\therefore$ $f(x)$ is differentiable on $\left(-\frac{1}{2}, \sqrt{2}\right)$

(3) $f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) - 2$

$$= -\frac{1}{4} + \frac{1}{4} + 2 - 2 = 0$$

$$f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4(\sqrt{2}) - 2$$

$$= 4(\sqrt{2}) + 2 - 4\sqrt{2} - 2 = 0$$

$\therefore$ $f\left(-\frac{1}{2}\right) = f(\sqrt{2})$

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's Theorem on $\left[-\frac{1}{2}, \sqrt{2}\right]$.

$\therefore$ there exists atleast one $c \in \left(-\frac{1}{2}, \sqrt{2}\right)$

such that $f'(c) = 0$

$\Rightarrow$ $6c^2 + 2c - 4 = 0$

$\Rightarrow$ $3c^2 + c - 2 = 0$

$\Rightarrow$ $3c^2 + 3c - 2c - 2 = 0$

$\Rightarrow$ $3c(c + 1) - 2(c + 1) = 0$

$\Rightarrow$ $(c + 1)(3c - 2) = 0$

$\Rightarrow$ $c = -1, \frac{2}{3}$

we see that

$$\frac{2}{3} \in \left(-\frac{1}{2}, \sqrt{2}\right)$$

$\therefore$ Rolle's Theorem is verified.

Q. 2. If $y = (\sin x)^{(\sin x)^{\cdot^{\cdot^{\infty}}}}$, then prove that

$$\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log_e \sin x}$$ **(*BSER*, 2013)**

Solution :

$$y = (\sin x)^{(\sin x)^{\cdot^{\cdot^{\infty}}}}$$

$\Rightarrow$ $y = (\sin x)^y$

$\Rightarrow$ $\log y = y \log \sin x$

(Taking log on both sides)

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = y \cdot \frac{d}{dx}(\log_e \sin x) + \log \sin x \frac{dy}{dx}$$

$\Rightarrow$ $\frac{1}{y}\frac{dy}{dx} = y \frac{1}{\sin x} \cdot \cos x + \log \sin_e x \frac{dy}{dx}$

$\Rightarrow$ $\frac{dy}{dx}\left(\frac{1}{y} - \log_e \sin x\right) = y \cot x$

$\Rightarrow$ $\frac{dy}{dx}\left(\frac{1 - y \log_e \sin x}{y}\right) = y \cot x$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y \log_e \sin x}$

Q. 3. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5 & , \text{ if } x \le 2 \\ ax + b & , \text{ if } 2 < x < 10 \\ 21 & , \text{ if } x \ge 10 \end{cases}$$

is a continuous function. (*BSER*, 2014)

Solution :

$f(2) = 5$ ...(1)

L.H.L. $= \lim_{x \to 2^-} f(x)$

$= \lim_{h \to 0} f(2 - h)$

$= \lim_{h \to 0} 5$

$= 5$ ...(2)

R.H.L. $= \lim_{h \to 2^+} f(x)$

$= \lim_{h \to 0} f(2 + h)$

$= \lim_{h \to 0} \{a(2 + h) + b\}$

$= 2a + b$ ...(3)

$\because$ $f(x)$ is a continuous function

$\therefore$ $f(x)$ is continuous at $x = 2$

$\therefore$ L.H.L. = R.H.L. = $f(2)$

$\Rightarrow$ $5 = 2a + b = 5$

$\Rightarrow$ $2a + b = 5$ ...(4)

Again, $f(10) = 21$ ...(5)

L.H.L. $= \lim_{x \to 10^-} f(x)$

$= \lim_{h \to 0} f(10 - h)$

$= \lim_{h \to 0} \{a(10 - h) + b\}$

$= 10a + b$ ...(6)

R.H.L. $= \lim_{x \to 10^+} f(x)$

$= \lim_{h \to 0} f(10 + h)$

$= \lim_{h \to 0} 21$

$= 21$ ...(7)

$\because$ $f(x)$ is a continuous function

$\therefore$ $f(x)$ is continuous at $x = 10$

$\therefore$ L.H.L. = R.H.L. = $f(10)$

$\Rightarrow$ $10a + b = 21 = 21$

$\Rightarrow$ $10a + b = 21$ ...(8)

Solving (4) and (5), we get

$a = 2, b = 1$

Q. 4. If the following function $f(x)$ is continuous at $x = 0$, find the value of a. (*JAC*, 2014)

$$f(x) = \begin{cases} \frac{1 - \cos 4x}{x^2} & ; \ x < 0 \\ a & ; \ x = 0 \\ \frac{\sqrt{x}}{\sqrt{16 + x}} & ; \ x > 0 \end{cases}$$

Solution :

$f(o) = a$ ...(1)

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0} \frac{1-\cos 4(-h)}{(-h)^2}$$
$$= \lim_{h\to 0} \frac{1-\cos 4h}{h^2}$$
$$= \lim_{h\to 0} \frac{2\sin^2 2h}{h^2}$$
$$= \lim_{h\to 0} 2\left(\frac{\sin 2h}{2h}\right).4$$
$$= 8 \qquad \ldots(2)$$
$$\text{R.H.L.} = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0} \frac{\sqrt{h}}{\sqrt{16+\sqrt{h}-4}}$$
$$= \lim_{h\to 0} \frac{\sqrt{h}\left\{\sqrt{16-\sqrt{h}}+4\right\}}{\sqrt{16+\sqrt{h}}-4(\sqrt{16+\sqrt{h}}+4)}$$
$$= \lim_{h\to 0} \frac{\sqrt{h}\left\{\sqrt{16+\sqrt{h}}+4\right\}}{16+\sqrt{h}-16}$$
$$= \lim_{h\to 0} (\sqrt{16+\sqrt{h}}+4)$$
$$= 4+4$$
$$= 8 \qquad \ldots(3)$$

If $f(x)$ is continuous at $x = 0$, then

$$\text{L.H.L.} = \text{R.H.L.} = f(0)$$
$$\Rightarrow \quad 8 = 8 = a$$
$$\Rightarrow \quad 8 = a$$
$$\Rightarrow \quad a = 8$$

Q. 5. Examine the function

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

for its continuity at $x = 0$. (*USEB*, 2013)

Solution :

$$f(0) = 0 \qquad \ldots(1)$$
$$\text{L.H.L.} = \lim_{x\to 0^-} f(x)$$
$$= \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} f(-h)$$
$$= \lim_{h\to 0} -(-h)$$
$$= \lim_{h\to 0} h = 0 \qquad \ldots(2)$$
$$\text{R.H.L.} = \lim_{x\to 0^+} f(x)$$
$$= \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0} h$$
$$= 0 \qquad \ldots(3)$$

$\because$ L.H.L. = R.H.L. = $f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Q. 6. Write mean value theorem and verify it for the function $f(x) = x^2$ in the interval [2, 4]. (*USEB*, 2014)

Solution :

We have $f(x) = x^2$

(1) $\because$ $f(x)$ is a polynomial

$\therefore$ $f(x)$ is continuous on R.

$\therefore$ In particular, $f(x)$ is continuous on [2, 4].

(2) $f'(x) = 2x$

$\Rightarrow$ $f'(x)$ exists uniquely on (2, 4).

$\Rightarrow$ $f(x)$ is derivable on (2, 4).

$\Rightarrow$ $f(x)$ satisfies the condition of Lagrange's mean value theorem on [2, 4].

$\therefore$ These exists atleast one $c \in (2, 4)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c), \text{ where } a = 2, b = 4$$
$$\Rightarrow \frac{f(4)-f(2)}{4-2} = 2c$$
$$\Rightarrow \frac{4^2-2^2}{2} = 2c$$
$$\Rightarrow 6 = 2c$$
$$\Rightarrow c = 3 \in (2, 4)$$

$\therefore$ Lagrange's mean value theorem is verified.

Q. 7. Show that the function $f(x) = |x-3|, x \in \mathbb{R}$ is continuous but not differentiable at $x = 3$. (*CBSE*, 2013)

Solution :

$$f(3) = |3-3| = 0 \qquad \ldots(1)$$
$$\text{L.H.L.} = \lim_{x\to 3^-} f(x)$$
$$= \lim_{h\to 0} f(3-h)$$
$$= \lim_{h\to 0} |3-h-3|$$
$$= \lim_{h\to 0} |-h|$$
$$= 0 \qquad \ldots(2)$$
$$\text{R.H.L.} = \lim_{x\to 3^+} f(x)$$
$$= \lim_{h\to 0} f(3+h)$$
$$= \lim_{h\to 0} |3+h-3|$$
$$= \lim_{h\to 0} |h|$$
$$= 0 \qquad \ldots(3)$$

$\because$ L.H.L. = R.H.L. = $f(0)$

$\therefore$ $f(x)$ is continuous at $x = 3$.

$$\text{L.H.D.} = \text{L}f'(3) = \lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3}$$
$$= \lim_{h\to 0} \frac{f(3-h)-f(3)}{3-h-3}$$
$$= \lim_{h\to 0} \frac{|3-h-3|-0}{-h}$$
$$= \lim_{h\to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$
$$= -1 \quad \text{...(4)}$$

$$\text{R.H.D.} = Rf'(3) = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3}$$
$$= \lim_{h \to 0} \frac{f(3+h) - f(3)}{3 + h - 3}$$
$$= \lim_{h \to 0} \frac{|3 + h - 3| - 0}{h}$$
$$= \lim_{h \to 0} \frac{h}{h}$$
$$= \lim_{h \to 0} 1 = 1 \quad \text{...(5)}$$

$\because$ L.H.D. $\neq$ R.H.D.

$\therefore$ $f(x)$ is not differentiable at $x = 3$.

Q. 8. If $\cos y = x \cos (a + y)$, show that :

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

(CBSE, AI, 2009; USEB, 2014)

Solution :

$$\cos y = x \cos (a + b)$$
$$\Rightarrow \quad x = \frac{\cos y}{\cos(a+y)}$$
$$\therefore \quad \frac{dy}{dx} = \frac{\cos(a+y).(-\sin y) - \cos y\{-\sin(a+y)\}}{\cos^2(a+y)}$$
$$= \frac{-\cos(a+y)\sin y + \sin\ (a+y)\cos y}{\cos^2(a+y)}$$
$$= \frac{\sin(a+y-y)}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Q. 9. Find the value of k, for which :

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \leq x < 0 \\ \dfrac{2x+1}{x-2}, & \text{if } \ 0 \leq x \leq 1 \end{cases}$$

is continuous at $x = 0$. *(AI CBSE, 2013)*

Solution :

$$f(0) = \frac{2(0)+1}{0-2} = -\frac{1}{2} \quad \text{...(1)}$$
$$\text{L.H.L.} = \lim_{x \to 0^-} f(x)$$
$$= \lim_{h \to 0} f(0 - h)$$
$$= \lim_{h \to 0} f(0 - h)$$
$$= \lim_{h \to 0} \frac{\sqrt{1+k(-h)} - \sqrt{1-k(-h)}}{-h}$$
$$= \lim_{h \to 0} \frac{\sqrt{1-kh} - \sqrt{1+kh}}{-h}$$
$$= \lim_{h \to 0} \frac{(\sqrt{1-kh} - \sqrt{1+kh})(\sqrt{1-kh} + \sqrt{1+kh})}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$$
$$= \lim_{h \to 0} \frac{(1-kh)-(1+kh)}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$$
$$= \lim_{h \to 0} \frac{2k}{\sqrt{1-kh} + \sqrt{1+kh}}$$
$$= \frac{2k}{\sqrt{1-0} + \sqrt{1+0}} = k \quad \text{...(2)}$$
$$\text{R.H.L.} = \lim_{x \to 0^+} f(x)$$
$$= \lim_{h \to 0} f(0 + h)$$
$$= \lim_{h \to 0} f(h)$$
$$= \lim_{h \to 0} \frac{2h+1}{h-2}$$
$$= \frac{2(0)+1}{0-2} = -\frac{1}{2} \quad \text{...(3)}$$

If $f(x)$ is continuous at $x = 0$, then

L.H.L. = R.H.L. = $f(0)$

$$\Rightarrow \quad k = -\frac{1}{2} = -\frac{1}{2}$$
$$\Rightarrow \quad k = -\frac{1}{2}$$

Q. 10. If $f(x) = \begin{cases} 1, & \text{if } \ x \leq 3 \\ ax + b, & \text{if } \ 3 < x < 5, \\ 7, & \text{if } \ x \geq 5 \end{cases}$

find the values of a and b, so that $f(x)$ is a continuous function. *[CBSE, 2013 (Comptt.)]*

Solution :

$$f(3) = 1 \quad \text{...(1)}$$
$$\text{L.H.L.} = \lim_{x \to 3^-} f(x)$$
$$= \lim_{h \to 0} f(3 - h)$$
$$= \lim_{h \to 0} 1 = 1 \quad \text{...(2)}$$
$$\text{R.H.L.} = \lim_{x \to 3^+} f(x)$$
$$= \lim_{h \to 0} f(3 + h)$$
$$= \lim_{h \to 0} \{a\,(3 + h) + b\}$$
$$= 3a + b \quad \text{...(3)}$$

$\because$ $f(x)$ is a continuous function

$\therefore$ $f(x)$ is continuous at $x = 3$

$$\therefore \quad \text{L.H.L.} = \text{R.H.L.} = f(3)$$
$$\Rightarrow \quad 1 = 1 = 3a + b$$
$$\Rightarrow \quad 1 = 3a + b$$
$$\Rightarrow \quad 3a + b = 1 \quad \text{...(4)}$$
$$f(5) = 7 \quad \text{...(5)}$$
$$\text{L.H.L.} = \lim_{x \to 5^-} f(x)$$
$$= \lim_{h \to 0} f(5 - h)$$
$$= \lim_{h \to 0} \{a\,(5 - h) + b\}$$
$$= 5a + b \quad \text{...(6)}$$
$$\text{R.H.L.} = \lim_{x \to 5^+} f(x)$$
$$= \lim_{h \to 0} f(5 + h)$$

$$= \lim_{h\to 0} 7 = 7 \quad ...(7)$$

$\because f(x)$ is a continuous function

$\therefore f(x)$ is continuous at $x = 5$

$\therefore \quad$ L.H.L. = R.H.L. = $f(5)$

$\Rightarrow \quad 5a + b = 7 = 7$

$\Rightarrow \quad 5a + b = 7 \quad ...(8)$

solving equations (4) and (8), we get

$$a = 3, b = -8$$

Q. 11. Find the value of the constant k so that the function f, defined below, is continuous at $x = 0$, where

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

[*AI CBSE*, 2014 (*Comptt.*)]

Solution :

$$f(0) = k \quad ...(1)$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \left\{\frac{1-\cos 4(-h)}{8(-h)^2}\right\}$$

$$= \lim_{h\to 0} \frac{1-\cos 4h}{8h^2}$$

$$= \lim_{h\to 0} \frac{2\sin^2 2h}{8h^2}$$

$$= \lim_{h\to 0} \frac{\sin^2 2h}{4h^2}$$

$$= \lim_{h\to 0} \left(\frac{\sin 2h}{2h}\right)^2$$

$$= (1)^2 = 1 \quad ...(2)$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x)$$

$$= \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{1-\cos 4h}{8h^2}$$

$$= 1 \quad ...(3)$$

If $f(x)$ is continuous at $x = 0$, then

$$\text{L.H.L.} = \text{R.H.L.} = f(0)$$

$\Rightarrow \quad 1 = 1 = k$

$\Rightarrow \quad k = 1$

Q. 12. In the interval [1, 2], find C of mean value theorem for the function $f(x) = 2x^2 - 1$. (*USEB*, 2015)

Solution : We have $f(x) = 2x^2 - 1$

$\because f(x)$ is a polynomial

$\therefore f(x)$ is continuous on [1, 2]

and $f(x)$ is derivable on [1, 2]

$\therefore f(x)$ satisfies the condition of langrange's mean value theorem on [1, 2]

$\therefore$ These exists at least one $C \in (1, 2)$ such that

$$f'(c) = \frac{f(2)-f(1)}{2-1} \quad ...(1)$$

$\because \quad f(2) = 2 \times (2)^2 - 1 = 8 - 1 = 7$

and $\quad f(1) = 2 \times (1)^2 - 1 = 2 - 1 = 1$

from equation (1),

$$f'(C) = \frac{7-1}{1}$$

$\Rightarrow \quad 2C = 6$

$\Rightarrow \quad C = \dfrac{6}{2}$

$\Rightarrow \quad C = 3$

Q. 13. Verify Rolle's Theorem for the function $f(x) = x^2 - 4x + 3$ in the interval [1, 3]. (*JAC*, 2015)

Solution : Given function $f(x) = x^2 - 4x + 3$ is a polynomial, $f(x)$ is continuous on (1, 3)

Now, $\quad f'(x) = 2x - 4 \;\forall\; x \in (1, 3)$ all exists

$\therefore f(x)$ is differentiable on (1, 3).

Again, $\quad f(1) = (1)^2 - 4 \times 1 + 3 = 1 - 4 + 3 = 0$

and $\quad f(3) = (3)^2 - 4 \times 3 + 3 = 9 - 12 + 3 = 0$

$\therefore \quad f(1) = f(3) = 0$

Hence, A point $C \in (1, 3)$ exists such that

$$f'(C) = 2C - 4 = 0$$

$\Rightarrow \quad 2C = 4$

$\Rightarrow \quad C = 2$

$\therefore \quad C = 2 \in (1, 3)$ such that $f'(C) = 0$

Hence, it verify Rolle's Theorem.

Q. 14. If $y = \tan^{-1} x$, then find $\dfrac{dy}{dx}$ by first principle. (*BSEB*, 2015)

Solution : $y = \tan^{-1} x$, then $x = \tan y$

Again, Let Δx is a small difference and Δy is also small difference.

$$x + \Delta x = \tan(y + \Delta y)$$

$\Rightarrow \quad \Delta x = \tan(y + \Delta y) - x$

$\Rightarrow \quad \Delta x = \tan(y + \Delta y) - \tan y$

$$\therefore \quad \frac{\Delta x}{\Delta y} = \frac{\tan(y+\Delta y) - \tan y}{\Delta y}$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\tan(y+\Delta y) - \tan y}$$

$$\Rightarrow \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta y\to 0} \frac{\Delta y}{\tan(y+\Delta y) - \tan y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\Delta y\to 0} \frac{dy}{\dfrac{\sin(y+\Delta y)}{\cos(y+\Delta y)} - \dfrac{\sin y}{\cos y}}$$

$$= \lim_{\Delta y\to 0} \frac{\Delta y.\cos y.\cos(y+\Delta y)}{\sin(y+\Delta y)\cos y - \cos(y+\Delta y)\sin y}$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y}{\sin(y + \Delta y - y)} \cos y . \cos(y + \Delta y)$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y}{\sin \Delta y} \lim_{\Delta y \to 0} \cos y . \cos(y + \Delta y)$$

$$= \cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y}$$

$$= \frac{1}{1 + x^2}$$

NCERT QUESTIONS

Q. 1. Find the value of '*a*' for which the function *f* defined as :

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

is continuous at $x = 0$. (*CBSE*, 2011)

Solution :

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

∵ Given function is continuous at $x = 0$,

∴ L.H.L. = R.H.L. = $f(0)$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(0 + h) = f(0) \quad ...(1)$$

Finding L.H.L.,

$$\lim_{h \to 0} f(0 - h) = \lim_{h \to 0} a \sin \frac{\pi}{2}(-h + 1)$$

$$= a \lim_{h \to 0} \sin\left(\frac{\pi}{2} - \frac{\pi h}{2}\right)$$

$$= a \times 1 = a \quad ...(2)$$

$$\text{R.H.L.} = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} \frac{\tan h - \sin h}{h^3}$$

$$= \lim_{h \to 0} \frac{\sin h(1 - \cos h)}{\cos h \times h^3}$$

$$= \left(\lim_{h \to 0} \frac{\sin h}{h}\right) \times \left(\lim_{h \to 0} \frac{1 - 1 + 2\sin^2 \frac{h}{2}}{h^2}\right)$$

$$= 1 \times 2\left(\lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}}\right) \times \frac{1}{4} = \frac{1}{2} \quad ...(3)$$

From equations (2), (3) and (4), we get

$$f(0) = a \sin \frac{\pi}{2} (0 + 1) = a \quad ...(4)$$

$$\Rightarrow \quad a = \frac{1}{2}$$

Q. 2. Differentiate : $x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$ w.r.t. x. (*CBSE*, 2011)

Solution :

Let $y = x^{x \cos x} + \frac{x^2 + 1}{x^2 - 1}$

⇒ Let $y_1 = x^{x \cos x}$

Taking log on both sides,

$$\log y_1 = (x \cos x) \log x \quad ...(1)$$

By differentiating,

$$\frac{1}{y_1}\frac{dy_1}{dx} = \frac{(x \cos x)}{x} + (\log x)\{-x \sin x + \cos x\}$$

$$\Rightarrow \frac{dy_1}{dx} = x^{x \cos x}[\cos x + (\log x)(\cos x - x \sin x)] \quad ...(2)$$

Let $y_2 = \frac{x^2 + 1}{x^2 - 1}$

∴ By differentiating,

$$\frac{dy_2}{dx} = \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2}$$

$$\Rightarrow \frac{dy_2}{dx} = \frac{-4x}{(x^2 - 1)^2} \quad ...(3)$$

From equations (2) and (3),

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

$$\therefore \frac{dy}{dx} = x^{x \cos x}[\cos x + (\log x)(\cos x - x \sin x) - \frac{4x}{(x^2 - 1)^2}]$$

Q. 3. If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$

show that : $\frac{dy}{dx} = -\frac{y}{x}$. (*CBSE, Outside Delhi*, 2012)

Solution :

$$xy = \sqrt{a^{\sin^{-1} t}}\sqrt{a^{\cos^{-1} t}}$$

$$= \sqrt{a^{\sin^{-1} t + \cos^{-1} t}}$$

$$= \sqrt{a^{\pi/2}}$$

Differentiating with respect to x, we get

$$x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

■■